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Representations of Codimension $\geqslant 3$ by Definite Quadratic Forms

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Let M be a positive definite quadratic $\mathbf{Z}$-lattice of rank $\geqslant n+3$. If N is a quadratic $\mathbf{Z}$-lattice of rank n which is primitively represented by the genus of M and if all the successive minima of N increase sufficiently quickly, then there exists a global primitive representation of N by M with approximation and primitivity properties.

1. INTRODUCTION

Let M be a positive definite quadratic $\mathbf{Z}$-lattice of rank m, and N a quadratic $\mathbf{Z}$-lattice of rank n which is primitively represented by M locally. Theorem 2.5 in [3] shows that there exists a global representation of N by M with approximation and primitivity properties provided $m \geqslant 3n+3$ and the minimum of N is sufficiently large. Recent work by Kitaoka and Jöchner [4, 5] suggests that $m \geqslant 2n+3$ may suffice. This is the case when $n=1$ (see the remark at the end of [5]) and $n=2$ [4]. When $n=1$, one can even push down to $m \geqslant 4$, see Theorem 2.1 in [2]. In this article, we generalize the result to the case where n is arbitrary and $m \geqslant n+3$ by imposing conditions on the sizes of the successive minima of N. For $i=1, ..., n$, let $S_i(M, n)$ be the set of positive rational numbers which are the ith successive minima of rank n $\mathbf{Z}$-lattices primitively represented by the genus of M. The following is our main theorem.

THEOREM 1.1. *Let n be a positive integer and M a positive definite quadratic $\mathbf{Z}$-lattice of rank $\geqslant n+3$. Let s be a positive integer and let T be a*

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finite set of primes. There exist real valued functions $c_1 = c_1(M, T, s)$, $c_2 = c_2(M, T, s, \lambda_1)$, ..., $c_n = c_n(M, T, s, \lambda_{n-1})$, $\lambda_i \in S_i(M, n)$, *with the following property*:

Let N *be a positive definite rank* n **Z**-*lattice with successive minima* $\mu_1 \leqslant \cdots \mu_n$ *and* $f_p\colon N_p \to M$ *primitive representations at each prime* p. *If* $\mu_1 \geqslant c_i(M, T, s, \mu_{i-1})$ *for* $i = 1, ..., n$, *then there exists a global representation* $f\colon N \to M$ *satisfying the conditions*

1. $f \equiv f_p \bmod p^s M_p$ *for all* $p \in T$,
2. $f(N)$ *is primitive in* M,

We refer the reader to [1] and [7] for any undefined terminology and notation. In particular, $s(M)$ denotes the scale of M. All lattices and the associated quadratic forms are understood to be positive definite. Given a nonzero vector w in the ambient space V of M, let $\Phi_w\colon V \to V$ be the projection $\Phi_w(x) = x - ((w, x)/(w, w))w$ onto the orthogonal complement of w, where (,) is the bilinear form on M. For any $w \in M$, we define $\gamma_w = Q(w)/s(M)$. Then γ_w is an ideal of **Z**. By abuse of notation, we also use γ_w to denote a generator of this ideal. Note that $\gamma_w \Phi_w(N) \subseteq N$ for any submodule N of M containing w. Linearly independent vectors $v_1, ..., v_n \in V$ are Hermite reduced if the corresponding quadratic form $f(x_1, ..., x_n) = Q(x_1 v_1 + \cdots + x_n v_n)$ is Hermite reduced. We refer the reader to Chapter 12 of [1] for the facts needed below for Hermite reduced forms and Siegel domains. In particular, Hermite reduced is defined inductively by $Q(v_1)$ is the minimum nonzero norm on $N = \mathbf{Z}[v_1, ..., v_1]$, $|2(v_1, v_i)| \leqslant Q(v_1)$, and $\Phi_{v_1}(v_2), ..., \Phi_{v_1}(v_n)$ is Hermite reduced. Any vector in N of minimal nonzero norm can be extended to a Hermite reduced basis for N and the form $f(x_1, ..., x_n) = Q(x_1 v_1 + \cdots + x_n v_n)$ for a Hermite reduced basis lies in the Siegel domain $\mathscr{S}_n(4/3, 1/2)$.

LEMMA 1.1. *Let* v *be a vector in* N *of minimal nonzero norm. Let* $\mu_1, ..., \mu_n$ *denote the successive minima for* N *and* $\mu'_1, ..., \mu'_{n-1}$ *the successive minima for* $N' = \Phi_v(N)$. *Then there is a positive constant* $C = C(n)$ *such that* $(\mu_{i+1})/\mu_i$, $(\mu'_i)/\mu_{i+1} \leqslant C$ *for* $i = 1, ..., n-1$.

Proof. Let $v = v_1, v_2, ..., v_n$ be a Hermite reduced basis for N. By completing squares, one obtains $f(x_1, ..., x_n) = Q(x_1 v_1 + \cdots + x_n v_n) = h_1(x_1 + c_{12} x_2 + \cdots + c_{1n} x_n)^2 + g(x_2, ..., x_n)$ where $g(x_2, ..., x_n) = Q(x_2 \Phi_v(v_2) + \cdots + x_n \Phi_v(v_n)) = h_2(x_2 + c_{23} x_3 + \cdots + c_{2n} x_n)^2 + \cdots + h_n x_n^2$. The quadratic forms f and g are Hermite reduced and therefore are in the Siegel domains $\mathscr{S}_n(4/3, 1/2)$ and $\mathscr{S}_{n-1}(4/3, 1/2)$ respectively. Thus the quotients h_j/μ_j, μ_j/h_j, $j = 1, ..., n$, and h_{j+1}/μ'_j, μ'_j/h_{j+1}, $j = 1, ..., n-1$ are bounded by a constant depending only on n, and the lemma follows (see, e.g., [1], page 266).

2. PROOF OF MAIN RESULT

We can extend T by adjoining a finite set of primes depending only on M so that for any $p \notin T$, p is odd and M_p is unimodular. We proceed with induction on n. We first make some preliminary adjustments to facilitate the proof. Choose a positive integer $e = e(M, s)$ so that the following holds at $p \in T$ ([6], Cor 5.4.3):

> $(*)$ if x and y are primitive vectors in M_p satisfying $x \equiv y \bmod p^e M_p$ and $Q(x) = Q(y)$ then there exists $\phi_p \in O^+(M_p)$ with $\phi_p(x) = y$ and $\phi_p \equiv 1 \bmod p^s M_p$.

We now define the functions $c_i(M, T, s, \lambda_{i-1})$ by induction on n. The case $n = 1$ is covered by Theorem 2.1 of [2] (for $m = 4$) and [5] (for $m \geqslant 5$). Note that the statement of Theorem 2.1 of [2] requires M to be integral. However, by scaling M suitably, the theorem is then applicable and we may assume $n > 1$. For $i = 1$, we define $c_1(M, T, s)$ to be the constant obtained by applying the one dimensional result to M, T, e. For $n \geqslant i \geqslant 2$, we fix $\lambda_{i-1} \in S_{i-1}(M, n)$. Then there exists a rank n $\mathbf{Z}$-lattice L primitively represented by the genus of M having $i-1$th successive minimum λ_{i-1}. Let λ_1 be the first successive minimum of L and note that since $\lambda_1 \leqslant \lambda_{i-1}$, there are only finitely many choices for λ_1 (although there are possibly infinitely many choices for L). If no such choice provides a λ_1 primitively represented by M, then we define $c_i(M, T, s, \lambda_{i-1})$ to be zero. For each λ_1 primitively represented by M, we pick a primitive vector $w \in M$ such that $Q(w) = \lambda_1$. Again, there are only finitely many choices for w. Set $M' = \Phi_w(M)$, $T' = T \cup \{p\colon ord_p(Q(w)) \neq 0\}$ and s' any integer satisfying $s' \geqslant s + ord_p(\gamma_w)$. Let $c_1(M', T', s')$ and $c_{i-1}(M', T', s', \lambda'_{i-2})$ be the functions given by the induction hypothesis for the case M', T', s', $n-1$. Now for $i = 2$, we set

$$c_2(M, T, s, \lambda_1) = C \max\{c_1(M', T', s')\}$$

where C is the constant given by Lemma 1.1 and the maximum is taking over all choices of M', T', s'. For $i \geqslant 3$, define

$$c_i(M, T, s, \lambda_{i-1}) = C \max\{c_{i-1}(M', T', s', \lambda'_{i-2})\}.$$

Here the maximum is taking over all choices of M', T', s', $\lambda'_{i-2} \in S_{i-2}(M', n-1)$, $\lambda'_{i-2} \leqslant C\lambda_{i-1}$. There are only finitely many such λ'_{i-2} since $S_{i-2}(M', n-1) \subseteq s(M')$.

To complete the proof, let N, f_p be as given in the statement of the theorem. Let $\mu_1 \leqslant \cdots \leqslant \mu_n$ be the successive minima of N. Suppose that

$\mu_i \geqslant c_1(M, T, s)$ and $\mu_i \geqslant c_i(M, T, s, \mu_{i-1})$ for $i \geqslant 2$. Let $v \in N$ be a vector of norm μ_1. Then there exists a primitive vector w in M such that $Q(w) = \mu_1$ and $w \equiv f_p(v) \bmod p^e M_p$ for $p \in T$. We now adjust the local representation f_p. If $p \notin T$, then p is odd and M_p is unimodular. As the vectors $f_p(v)$ and w are both primitive in M_p and have the same norm, there exists $\phi_p \in O^+(M_p)$ with $\phi_p(f_p(v)) = w$. If $p \in T$, we also have $\phi_p \in O^+(M_p)$ with $\phi_p(f_p(v)) = w$ according to $(*)$. We can now assume that $f_p(v) = w$ for all p.

We next pass to the projections $M' = \Phi_w(M)$, $N' = \Phi_v(N)$. Define $f'_p \colon N'_p \to M'_p$ by

$$(**) \quad f'_p(\Phi_v(u)) = \Phi_w(f_p(u)).$$

It is easy to see that f'_p is a representation. Moreover, $f'_p(N'_p)$ is primitive in M'_p. For, if $t\Phi_w(y) \in f'_p(N'_p)$ with $t \in \mathbf{Z}_p$, $y \in M_p$, then $\Phi_w(\gamma_w t y) \in f_p(N_p)$ since $\gamma_w f'_p(N'_p) = \gamma_w \Phi_w(f_p(N_p)) \subseteq f_p(N_p)$. This shows that $\gamma_w t y \in f_p(N_p)$. The primitivity of $f_p(N_p)$ implies $y \in f_p(N_p)$ and hence $\Phi_w(y) \in \Phi_w(f_p(N_p)) = f'_p(N'_p)$.

Let $\mu'_1 \leqslant \cdots \leqslant \mu'_{n-1}$ be the successive minima of N'. Since $\mu_i \geqslant c_i$ and $C\mu_{i+1} \geqslant \mu'_i \geqslant C^{-1}\mu_{i+1}$, we have $\mu'_1 \geqslant c_1(M', T', s')$ and $\mu'_i \geqslant c_i(M', T', s', \mu'_{i-1})$ for $i = 2, ..., n-1$. By the induction hypothesis, we can find a representation $f' \colon N' \to M'$ satisfying conditions 1 and 2 of the theorem. Now we define $f \colon N \to V$ by

$$f(x) = f'(\Phi_v(x)) + \frac{(x, v)}{(v, v)}\, w.$$

Again, it is easy to see that f preserves the quadratic forms. Note also from $(**)$ and the induction hypothesis that for $x \in N$ and $p \in T'$, we have $f(x) - f_p(x) = f'(\Phi_v(x)) - f'_p(\Phi_v(x)) \in \gamma_w p^s M'_p \subseteq p^s M_p$. Thus, $f(x) \in M_p$ for all $p \in T'$. It is clear that $f(x) \in M_p$ for all $p \notin T'$. This shows that $f(x) \in M$ and condition 1 of the theorem is satisfied. Since $\mathbf{Z}_p v$ splits N_p and $\mathbf{Z}_p w$ splits M_p for $p \notin T'$, the primitivity of f at these p is a consequence of the fact that $f(v) = w$ and the primitivity of $f'(N')$. It follows that $f(N)$ is primitive in M and the theorem is proved.

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